EXERCISE – V HINTS & SOLUTIONS

Sol.1 $y^2 = x$ $\quad a = 1/4$

tangent to the parabola

$$y = mx + \frac{a}{m}$$

$$y = mx + \frac{1}{4m} \quad \ldots (1)$$

This tangent also touch the circle

$x^2 + y^2 - 6y + 4 = 0$

c (0, 3) $\quad r = \sqrt{5}$

p = r $\quad$ [for Eq^n (1)]

$$\left|\frac{3-\frac{1}{4m}}{\sqrt{1+m^2}}\right| = \sqrt{5}$$

$$\left(3-\frac{1}{4m}\right)^2 = 5(1+m^2)$$

After solving

$$m = \frac{1}{2} \qquad m = \frac{-5 \pm \sqrt{30}}{10}$$

So common tangents

$$y = mx + \frac{1}{4m} \quad \text{where } m = \frac{1}{2}$$

$$m = \frac{-5 \pm \sqrt{30}}{10}$$

Sol.2 (a)

$y^2 = kx - 8$

$$y^2 = k\left(x - \frac{8}{k}\right)$$

$$x - \frac{8}{k} = -\frac{k}{4}$$

$$x = \frac{8}{k} - \frac{k}{4} = 1$$

$32 - k^2 = 4k$

$k^2 + 4k - 32 = 0$

$k = -8$

$k = 4$

(b) $y = -x + k \quad \ldots (1)$

$y = mx - 2am - am^3 \quad a = 3$

$m = -1$ from Eq^n (1)

$y = -x + 2a + a \quad \ldots (2)$

(1) & (2) are same

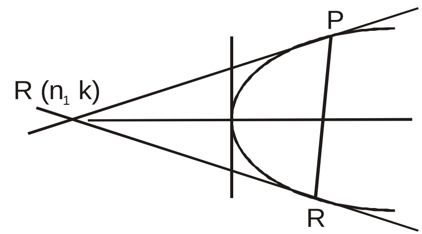
$k = 3a$

$k = 9 \quad (a = 3)$

Sol.3 $y^2 = 4ax$

Eq^n of PQ as normal to the parabola

$y = mx - 2am - am^3 \quad \ldots (1)$



Eq^n of PQ (c.o.c. w.r.t. R)

$ky = 2a(x + h) \quad \ldots (2)$

(1) & (2) are same

$$\frac{k}{1} = \frac{2a}{m} = \frac{2ah}{-2am - am^3}$$

$$m = \frac{2a}{k}$$

$$k = \frac{2ah}{-a(2m+m^3)}$$

$k(2m + m^3) = -2h$

$$k\left(\frac{4a}{k} + \frac{8a^3}{k^3}\right) = -2h$$

$$2a + \frac{4a^3}{k^2} = -h$$

$y^2(x + 2a) + 4a^3 = 0$

$y^2 = 8(x - 1) \Rightarrow a = 2$

Replace $x \to x - 1$

$y^2(x + 3) + 32 = 0$

Sol.4 (a) $y^2 = 4x$

Tangent to the parabola

$$y = mx + \frac{1}{m} \quad ; \quad a = 1$$

$m^2x - my + 1 = 0$

C.O.T for circle $\quad p = r$

c(3, 0), $r = 3$

$$\left|\frac{3m^2+1}{\sqrt{m^4+m^2}}\right| = 3$$

$(3m^2 + 1)^2 = 9(m^4 + m^2)$

$9m^4 + 6m^2 + 1 = 9m^4 + 9m^2$

$$3m^2 = 17 \qquad \Rightarrow \quad m = \pm\frac{1}{\sqrt{3}}$$

But above the axis

$$m = \frac{1}{\sqrt{3}}$$

$$y = \frac{x}{\sqrt{3}} + \sqrt{3}$$

$$\sqrt{3}y = x + 3$$

(b) $y^2 + 4y + 4 + 4x + 2 - 4 = 0$
$(y + 2)^2 + 4x - 2 = 0$
$(y + 2)^2 = -4x + 2$

$(y + 2)^2 = -4\left(x - \frac{1}{2}\right)$

$Y^2 = -4a X$ $\quad a = 1$
$X = a$

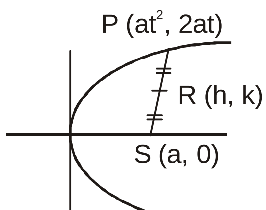
$x - \frac{1}{2} = 1 \Rightarrow x = \frac{3}{2}$

Sol.5 $h = \frac{at^2 + a}{2}$

$2h - a = at^2$ ... (1)

$k = \frac{2at + 0}{2}$

$t = \frac{k}{a}$ ... (2)



$2h - a = a\left(\frac{k^2}{a^2}\right)$

$k^2 = a(2h - a)$

$k^2 = 2a\left(h - \frac{a}{2}\right)$

$y^2 = 2a\left(x - \frac{a}{2}\right)$

Directrix $\quad x - \frac{a}{2} = \frac{-a}{2} \Rightarrow x = 0$

Sol.6 $y^2 = 8x$ $\quad xy = -1$
tangent to above parabola $\quad y = -1/x$

$y = mx + \frac{2}{m}$ ... (1)

$m^2x - my + 2 = 0$

$m^2x + \frac{m}{x} + 2 = 0$

$m^2x^2 + 2x + m = 0$
$D = 0$
$4 - 4m^3 = 0$
$m = 1$
from Eq^n (1)
$y = x + 2$

Sol.7 (a) $y^2 = 16x$ $\quad (a = 4)$
chord with two points
$2x - (t_1 + t_2)y + 2at_1t_2 = 0$ ... (1)
Passes through focus (4, 0)
$8 + 8t_1t_2 = 0 \Rightarrow t_1t_2 = -1$ ... (2)
Eq^n (1) is tangent to given circle

c (6, 0) ; $r = \sqrt{2}$

$$\left|\frac{12 + 2at_1t_2}{\sqrt{4 + (t_1 + t_2)^2}}\right| = \sqrt{2}$$

$16 = 2[4 + (t_1 + t_2)^2]$
$4 = (t_1 + t_2)^2$
$t_1 + t_2 = \pm 2$

Slope $= \frac{2}{t_1 + t_2} = \pm 1$

(b) Eq^n of Normal $\quad$ for $y^2 = 4x$
$y = mx - 2m - m^3$ ... (1)
(h, k)
$k = mh - 2m - m^3$
$m^3 + (2 - h)m + k = 0$ ... (2)
$m_1 m_2 m_3 = -k$

$am_3 = -k \Rightarrow m_3 = \frac{-k}{a}$

Now from Eq^n (2)

$$\frac{-k^3}{a^3} - (2 - h)\frac{k}{a} + k = 0$$

$k^3 + (2 - h)ka^2 - ka^3 = 0$
$y^2 + (2 - x)a^2 - a^3 = 0,$
as $y \neq 0$ (P does't le on the axis of parabola)
$y^2 = a^2x - 2a^2 + a^3$
If it is part of the parabola $y^2 = 4x$
then $a^2 = 4$ and $-2a^2 + a^3 = 0$
$\Rightarrow a = 2$

Sol.8 $y^2 = 4x$ $\quad a = 1$
POI of tangent
$[at_1t_2, a(t_1 + t_2)]$
$t_1t_2 = 1$
$t_1 + t_2 = 4$
General tangent $ty = x + at^2$

$m_1 = \frac{1}{t_1}$ $\quad m_2 = \frac{1}{t_2}$

$$\tan\theta = \left|\frac{\frac{1}{t_1} - \frac{1}{t_2}}{1 + \frac{1}{t_1t_2}}\right| = \left|\frac{t_1 - t_2}{1 + t_1t_2}\right| = \frac{\sqrt{(t_1 + t_2)^2 - 4t_1t_2}}{2}$$

$$= \frac{\sqrt{16 - 4}}{2}$$

$\tan\theta = \sqrt{3}$

$$\theta = \frac{\pi}{3}$$

Sol.9 $y^2 - 2y - 4x + 5 = 0$

$(y-1)^2 = 4(x-1)$

$a = 1$

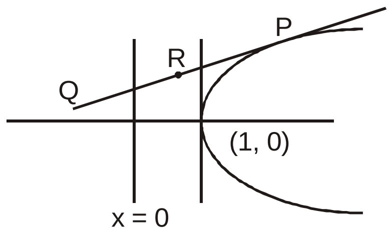
$Y^2 = 4X$

Let a point P on the parabola P $(t^2 + 1, 2t + 1)$

Eq^n of tangent

$t(y-1) = (x-1) + t^2$

$Q\left(0,\ 1 + t - \frac{1}{t}\right)$



R point devids the PQ in $\frac{1}{2}$: 1 that means P is the mid point of QR

$\frac{h+0}{2} = t^2 + 1$

$\Rightarrow t^2 = \frac{h}{2} - 1 = \left(\frac{h-2}{2}\right)$ … (1)

$1 + t - \frac{1}{t} + k = 2 + 4t$

$3t + \frac{1}{t} = k - 1$

Square both side

$9t^2 + \frac{1}{t^2} + 6 = (k-1)^2$

$$9\left(\frac{h-2}{2}\right) + \frac{1}{\left(\frac{h-2}{2}\right)} + 6 = (k-1)^2$$

$$9\left(\frac{h-2}{2}\right)^2 + 6\left(\frac{h-2}{2}\right) + 1 = (k-1)^2\left(\frac{h-2}{2}\right)$$

$9(h-2)^2 + 12(h-2) + 4 = 2(k-1)^2(h-2)$

$(3h-4)^2 = 2(k-1)^2(h-2)$

Locus : $(3x-4)^2 = 2(y-1)^2(x-2)$

Sol.10 (a) Eq^n of directrix $x + y = 0$

Hence Eq^n of the parabola is

$$\frac{x+y}{\sqrt{2}} = \sqrt{(x-2)^2 + (y-2)^2}$$

Eq^n of Parabola

$(x-y)^2 = 8(x+y-2)$

(b) Eq^n of tangent to $x^2 = y$ is

$y = mx - \frac{m^2}{4}$ … (1)

Eq^n of tangent to $(x-2)^2 = -y$ is

$y = m(x-2) + \frac{m^2}{4}$ … (2)

(1) & (2) are same

$m = 0$ or $m = 4$

$y = 0$ and $y = 4x - 4$

(c) Match the column

$y^2 = 4x$ $(a = 1)$

Normal $y = mx - 2m - m^3$

Passing through (3, 0)

$0 = 3m - 2m - m^3$

$m^3 = m$

$\Rightarrow m = 0$ & $m = \pm 1$

cantroid $= \left(\frac{\Sigma m_1^2}{3}, -\frac{2\Sigma m_1}{3}\right) = \left(\frac{2}{3}, 0\right)$

circum radius $= \left|\frac{-2m_1 + 2m_2}{2}\right| = 2$ units

Q : $(m_2^2, -2m_2) = (1, -2)$

R : $(m_3^2, -2m_3)$: (1, 2)

P(0, 0)

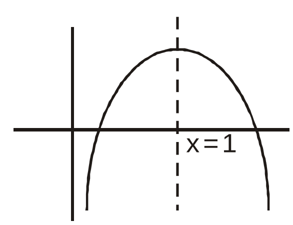
Area of $\Delta PQR = \frac{1}{2} \times 4 \times 1 = 2$ Sq. units

$$R = \frac{QR}{2\sin\angle QPR} = \frac{4}{2\sin(2\tan^{-1}2)}$$

$$= \frac{4}{2\sin\left(\tan^{-1}\frac{4}{1-4}\right)} = \frac{4}{2\times\frac{4}{5}} = \frac{5}{2}$$

circumcentre $\left(\frac{5}{2}, 0\right)$

Sol.11 $y = -\frac{x^2}{2} + x + 1$

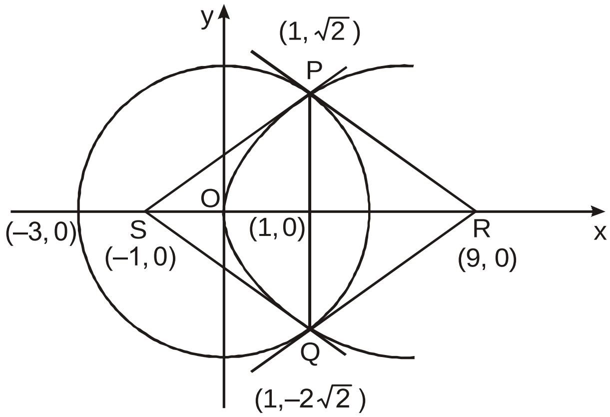


$y - \frac{3}{2} = -\frac{1}{2}(x-1)^2$

$(x-1)^2 = -2\left(y - \frac{3}{2}\right)$

symmetric about x = 1

Sol.12



(a) $P(1, \sqrt{2})$ $Q(1, -2\sqrt{2})$

Area of $\Delta PQR = \frac{1}{2}(4\sqrt{2})$ (8) = $16\sqrt{2}$

Area of $\Delta PQS = \frac{1}{2}(4\sqrt{2})$ (8) = $4\sqrt{2}$

Ratio of area of $\Delta PQS : \Delta PQR$ is 1 : 4

(b) Eq^n of circumcircle of ΔPRS

$(x + 1)(x - 9) + y^2 + \lambda y = 0$

It will passes through $(1, 2\sqrt{2}) \Rightarrow \lambda = 2\sqrt{2}$

Equation of circumcircle

$x^2 + y^2 - 8x + 2\sqrt{2}\,y - 9 = 0$

Hence radius = $3\sqrt{3}$

Aliter

Let $\angle PSR = \theta$

$$\sin\theta = \frac{2\sqrt{2}}{2\sqrt{3}}$$

PR = $6\sqrt{2}$ = 2R sin θ

R = $3\sqrt{3}$

(c) Radius of incircle in $r = \frac{\Delta}{S}$

$$\Delta = 16\sqrt{2}$$

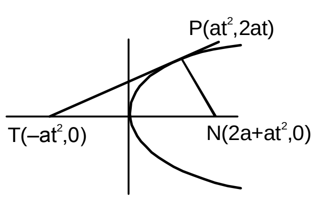
$$S = \frac{6\sqrt{2} + 6\sqrt{2} + 4\sqrt{2}}{2} = 8\sqrt{2}$$

$$r = \frac{16\sqrt{2}}{8\sqrt{2}} = 2$$

Sol.13 $G \equiv (h, k)$

$$h = \frac{2a + at^2}{3}$$

$$k = \frac{2at}{3}$$



$$\frac{3h - 2a}{a} = \frac{9k^2}{4a^2}$$

Required Parabola

$$\frac{9y^2}{4a^2} = \frac{3x - 2a}{a} = \frac{3}{a}\left(x - \frac{2a}{3}\right)$$

$$y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$$

Vertex : $\left(\frac{2a}{3}, 0\right)$

Focus : (a, 0)

Sol.14 $A(t_1^2, 2t_1)$, $B(t_2^2, 2t_2)$

Centre : $\left[\frac{t_1^2 + t_2^2}{2}, \; t_1 + t_2\right]$

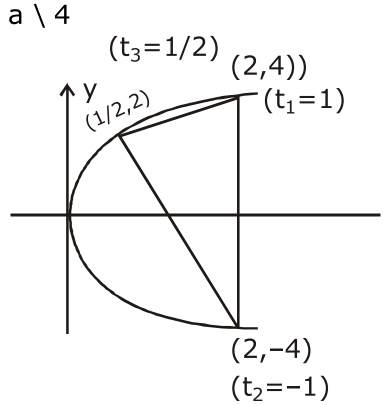
$t_1 + t_2 = \pm r$

slope of chord = $\frac{2}{t_1 + t_2} = \pm\frac{2}{r}$

Sol.15 for Δ_1 & Δ_2

$t_1 = 1$, $t_2 = -1$, $t_3 = 1/2$

a \ 4



$$\frac{\Delta_1}{\Delta_2} = \frac{1/2\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}}{1/2\begin{vmatrix} at_1t_2 & a(t_1+t_2) & 1 \\ at_2t_3 & a(t_2+t_3) & 1 \\ at_3t_1 & a(t_3+t_1) & 1 \end{vmatrix}} = \frac{24}{12} = 2$$

Sol.16 C

Let point P is (h, k)

$h = \frac{x}{4}$ & $k = \frac{y}{4}$

Locus of (h, k) is $y^2 = x$.